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Some Fixed Point Theorems via W-Distance on Cone Metric Spaces

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Some Fixed Point Theorems via w-Distance on Cone Metric Spaces

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Abstract - In this paper we present some fixed point theorems with the help of the concept of w- distance on cone metric spaces. Our results generalize and extend several well known results in the existing literature.

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I. Introduction and Preliminaries

In 1996, Kada et.al.[10] introduced the notion of w-distance on a metric space and proved a nonconvex minimization theorem which generalizes Caristi's fixed point theorem and the ϵ-variational principle. Afterwards, Huang and Zhang [8] initiated the notion of cone metric spaces by replacing the set of real numbers with an ordered Banach space . They also proved some fixed point theorems of contractive mappings on complete cone metric spaces with the assumption of normality of a cone. After that series of articles about cone metric spaces started to appear. In this work we extend the idea of w -distance on metric spaces to cone metric spaces and prove some fixed point theorems by considering w -distance on cone metric spaces. Our results generalize some recent results in fixed point theory.

Let E be a real Banach space and P be a subset of E Then P is called a cone if and only if

(i) P is closed; nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P\ \Rightarrow\ ax + by\ \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ (equivalently, $y \geq x$) if and only if $y - x \in P$. $x < y$ (equivalently, $y > x$) will stand for $x \leq y$ and $x \neq y$ while $x \ll y$ will stand for $y - x \in int\, P$, where $int\, P$ denotes the interior of P. For a finite subset A of E, if there exists an element $x \in A$ such that $x \leq a$ for all $a \in A$, we write $x = min\, A$. If there is an element $y \in A$ such that $a \leq y$ for all $a \in A$, we write $y = max\, A$. It is to be noted that $min\, A$, $max\, A$ are exist if the ordering $\leq$ on E is complete. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \ implies\ \ \| x \| \leq M \| y \| .$$

The least positive number satisfying the above inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0\ (n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Razapour and Hamlbarani [13] proved that there are no normal cones with normal constants $M < 1$ and for each $k > 1$ there are cones with normal constants $M > k$.

Definition 1.1. [8] Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies

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(i) $\theta \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.2. [8] Let (X,d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n) We denote this by $\lim_{n\to\infty} x_n = x \ or \ x_n \to x \ (n \to \infty)$.

Definition 1.3. [8] Let (X,d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n,m > n_0$, $d(x_n,x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Definition 1.4. [8] Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

We also note that the relations $int\,P + int\,P \subseteq int\,P$ and $\lambda\, int\,P \subseteq int\,P\,(\lambda > 0)$ hold.

Lemma 1.1. [14] Let (X,d) be a cone metric space and $a,b,c \in X$. Then

(i) If $a \ll b$ and $b \ll c$ then $a \ll c$.

(ii) If $a \leq b$ and $b \ll c$ then $a \ll c$.

Here we present some elementary results of [8].

Let (X,d) be a cone metric space, P a normal cone with normal constant M, $x \in X$ and (x_n) a sequence in X.Then

(i) (x_n) converges to x if and only if $d(x_n,x) \to \theta$ (Lemma 1).

(ii) Limit point of every sequence is unique (Lemma 2).

(iii) Every convergent sequen ce is Cauchy (Lemma 3).

(iv) (x_n) is a Cauchy sequence if and only if $d(x_n,x_m) \to \theta$ as $n,m \to \infty$ (Lemma 4).

(v) If $x_n \to x$ and $y_n \to y$ then $d(x_n,y_n) \to d(x,y)$ as $n \to \infty$ (Lemma 5).

Proposition 1.1. [9] If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.

In the following definition we extend the idea of w-distance on metric spaces to cone metric spaces.

Definition 1.5. Let (X,d) be a cone metric space. Then a function $p : X \times X \to P$ is called a w-distance on X if the following conditions are satisfied:

(i) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,y,z \in X$;

(ii) for any $x \in X$, $p(x,.) : X \to P$ is lower semicontinuous $i.e.$, if $x \in X$,

$$y_n \to y \ \in X \ then \ p(x,y) \leq \lim_{n\to\infty} \ inf\, p(x,y_n);$$

(iii) for any $\theta \ll \alpha$, there exists $\theta \ll \beta$ such that $p(z,x) \ll \beta$ and $p(z,y) \ll \beta$ imply $d(x,y) \ll \alpha$.

Example 1.1. Let $E = R^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x,y)$ $(|\, x-y\,|, a\,|\, x-y\,|)$ where $a \geq 0$ is a constant. Then (X,d) is a cone metric space. We define $p : X \times X \to P$ by $p(x,y) = (c,c)$ for every $x,y \in X$, where c is a positive real number. Then p is a w - distance on X.

Proof. (i) and (ii) are obvious. To show (iii), for any $\theta \ll \alpha$, put $\beta = (\frac{c}{2}, \frac{c}{2})$. Then $p(z,x) \ll \beta$ and $p(z,y) \ll \beta$ imply $d(x,y) \ll \alpha$.

Example 1.2. Let (X,d) be a cone metric space, P a normal cone. Then d is a w-distance on X.
Proof. (i) and (ii) are obvious. To show (iii), let $0 \ll \alpha$ be given and put $\beta = \frac{\alpha}{2}$.Then if $d(z,x) \ll \beta$ and $d(z,y) \ll \beta$,we have

$$d(x,y) \leq d(z,x) + d(z,y) \ll \beta + \beta = \alpha.$$

Definition 1.6. Let (X,d) be a cone metric space. A mapping $T : X \to X$ is said to be expansive if there exists a real constant $c > 1$ satisfying $d(T(x),T(y)) \geq c\, d(x,y)$ for all $x,y \in X$.

2. Main Results

In this section we always suppose that E is a real Banach space, P is a non normal cone in E with $int\, P \neq \emptyset$ and $\leq$ is the partial ordering on E with respect to P.Throughout the paper we denote by N the set of all natural numbers.

We start with the following lemma that will be needed in the sequel.

Lemma 2.1. Let (X,d) be a cone metric space and let p be a w-distance on X.Let (x_n) and (y_n) be sequences in X. Let (α_n) and (β_n) be sequences in P converging to θ and let $x,y,z \in X$. Then the following hold:

(i) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$for any $n \in N$, then (y_n) converges to z;

(ii) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$for any $n \in N$, then $y = z$. In particular, if $p(x,y) = \theta$and $p(x,z) = \theta$, then $y = z$;

(iii) If $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in N$ with $m > n$,then (x_n) is a Cauchy sequence.

Proof. (i) Let $\theta \ll \alpha$ be given. Then there exists $\theta \ll \beta$ such that $p(u,v) \ll \beta$ and $p(u,z) \ll \beta$ imply $d(v,z) \ll \alpha$.Choose $n_0 \in N$ such that $\alpha_n \ll \beta$ and $\beta_n \ll \beta$ for every $n \geq n_0$.Now, for any $n \geq n_0, p(x_n,y_n) \leq \alpha_n \ll \beta$ and $p(x_n,z) \leq \beta_n \ll \beta$ and hence $d(y_n,z) \ll \alpha$.This implies that (y_n) converges to z.

It follows from (i) that (ii) holds.

To prove (iii), let $\theta \ll \alpha$ be given. As in the proof of (i), choose $\theta \ll \beta$ and then $n_0 \in N$ .Now for any $n,m \geq n_0 + 1,\ p(x_{n_0},x_n) \leq \alpha_{n_0} \ll \beta$ and $p(x_{n_0},x_m) \leq \alpha_{n_0} \ll \beta$ and hence $d(x_n,x_m) \ll \alpha$. This implies that (x_n) is a Cauchy sequence.

Theorem 2.1. Let (X,d) be a complete cone metric space with w-distance p and $\leq$ be a complete ordering on E with respect to P.Let T_1,T_2 be mappings from X into itself. Suppose that there exists $r \in [0,1)$ such that

$$max\,\{p(T_1(x),T_2T_1(x)),\ p(T_2(x),T_1T_2(x))\} \leq r\, min\,\{p(x,T_1(x)),p(x,T_2(x))\} \tag{2.1}$$

for every $x \in X$ and that

$$inf\,\{p(x,y) + min\,\{p(x,T_1(x)),\ p(x,T_2(x))\} : x \in X\} > \theta \tag{2.2}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2.Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$.Moreover, if $v = T_1(v) = T_2(v)$,then $p(v,v) = \theta$.

Proof. Let u_0 be an arbitrary element of X.A sequence (u_n) in X is defined by

$$\begin{aligned} u_n &= T_1(u_{n-1}), \text{if } n \text{ is odd} \\ &= T_2(u_{n-1}), \text{if } n \text{ is even.} \end{aligned}$$

Then applying condition (2.1), we have for any positive integer n,

$$p(u_n,u_{n+1}) \leq r\, p(u_{n-1},u_n). \tag{2.3}$$

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By repeated use of (2.3), we obtain

$$p(u_n, u_{n+1}) \le r^n\, p(u_0, u_1).$$

If $m > n$,then

$$\begin{aligned} p(u_n, u_m) &\le p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\le \left[r^n + r^{n+1} + \cdots + r^{m-1}\right] p(u_0, u_1) \\ &\le \frac{r^n}{1-r}\, p(u_0, u_1). \end{aligned}$$

Obviously, ($\frac{r^n}{1-r}\, p(u_0, u_1)$) is a sequence in P converging to θ.So, by Lemma 2.1(iii), (u_n) is a Cauchy sequence in X.Since X is complete, (u_n) converges to some point $z \in X$.Let $n \in N$ be fixed. Then since (u_m) converges to z and $p\,(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \le \lim_{m\to\infty} inf\, p(u_n, u_m) \le \frac{r^n}{1-r}\, p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2.Then by hypothesis

$$\begin{aligned} \theta &< inf\ \{p(x, z) + min\,\{p(x, T_1(x)), p(x, T_2(x))\} : x \in X\} \\ &\le inf\ \{p(u_n, z) + min\,\{p(u_n, T_1(u_n)), p(u_n, T_2(u_n))\} : n \in N\} \\ &\le inf\ \left\{\frac{r^n}{1-r}\, p(u_0, u_1) + p(u_n, u_{n+1}) : n \in N\right\} \\ &\le inf\ \left\{\frac{r^n}{1-r}\, p(u_0, u_1) + r^n\, p(u_0, u_1) : n \in N\right\} \\ &= \theta \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

Suppose that $v = T_1(v) = T_2(v)$ for some $v \in X$.Then

$$\begin{aligned} p(v, v) &= max\ \{p(T_1(v), T_2T_1(v)), p(T_2(v), T_1T_2(v))\} \\ &\le r\, min\ \{p(v, T_1(v)), p(v, T_2(v))\} \\ &= r\, min\ \{p(v, v), p(v, v)\} \\ &= r\, p(v, v). \end{aligned}$$

By Proposition 1.1, it follows that $p(v, v) = \theta$.

The following Corollary is the generalization of the result [10; Theorem 4] to cone metric spaces.

Corollary 2.1. Let (X , d) be a complete cone metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that

$$p(T(x), T^2(x)) \le r\, p(x, T(x))$$

for every $x \in X$ and that

$$inf\ \{p(x, y) + p(x, T(x)) : x \in X\} > \theta$$

for every $y \in X$ with $y \ne T(y)$.Then there exists $z \in X$ such that $z = T(z)$.Moreover, if $v = T(v)$,then $p(v, v) = \theta$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.1, the conclusion of the Corollary follows.

Note: It is worth mentioning that for the cases $T_1 = T_2$ it is suficient to assume that $\le$ is a partial ordering on

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E with respect to P instead of a complete ordering.

Using Corollary 2.1, we obtain the following theorem:

Theorem 2.2. Let (X, d) be a complete cone metric space, let p be a w-distance on X and let $T : X \to X$ be continuous. Suppose that there exists $r \in [0, 1)$ such that

$$p(T(x), T^2(x)) \leq r\, p(x, T(x))$$

for every $x \in X$ .Then there exists $z \in X$ such that $z = T(z)$.Moreover, if $v = T(v),$ then $p(v, v) = \theta.$

Proof. Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\{p(x, y) + p(x, T(x)) : x \in X\} = \theta.$$

Then, there is a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{p(x_n, y) + p(x_n, T(x_n))\} = \theta.$$

So, it must be the case that $p(x_n, y) \to \theta$ and $p(x_n, T(x_n)) \to \theta$. By Lemma 2.1(i),$(T(x_n))$converges to y.

Now,

$$\begin{aligned} p(x_n, T^2(x_n)) &\leq p(x_n, T(x_n)) + p(T(x_n), T^2(x_n)) \\ &\leq p(x_n, T(x_n)) + r\, p(x_n, T(x_n)) \\ &\longrightarrow \theta. \end{aligned}$$

Again, by Lemma 2.1(i), $(T^2(x_n))$ converges to y . Using continuity of T , we obtain

$$T(y) = T(\lim_n T(x_n)) = \lim_n T^2(x_n) = y$$

which is a contradiction.

Hence, if $y \neq T(y)$,then

$$inf\{p(x, y) + p(x, T(x)) : x \in X\} > \theta.$$

Now Corollary 2.1 applies to obtain the desired conclusion.

As an application of Corollary 2.1, we obtain the following results [8; Theorem 1; Theorem 3; Theorem 4].

Theorem 2.3. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(T(x), T(y)) \leq k\, d(x, y),\ \ for\ all\ x, y \in X, \tag{2.4}$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. Since P is normal, we treat d as a w- distance on X.From (2.4), it follows that

$$d(T(x), T^2(x)) \leq k\, d(x, T(x))\ \ for\ every\ x \in X.$$

Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\{d(x, y) + d(x, T(x)) : x \in X\} = \theta.$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(x_n, T(x_n))\} = \theta.$$

So, we have $d(x_n, y) \to \theta$ and $d(x_n, T(x_n)) \to \theta$.Then by Lemma 2.1(i), $(T(x_n))$ converges to y.Since P is normal, $d(T(x_n), T(y)) \to d(y, T(y))$ as $n \to \infty$.

By using (2.4), we have

$$d(T(x_n), T(y)) \leq k\, d(x_n, y) \text{ for any } n \in N.$$

Taking limit as $n \to \infty$, it follows that $d(y, T(y)) \leq \theta$ which implies that $-d(y, T(y)) \in P$. Also, $d(y, T(y)) \in P$ and hence $d(y, T(y)) = \theta$. So it must be the case that $y = T(y)$.

This is a contradiction.

Hence, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(x, T(x)) : x \in X\} > \theta.$$

Now Corollary 2.1 applies to obtain a fixed point of T. Clearly a fixed point of T is unique.

Theorem 2.4. Let (X, d) be a complete cone metric space, P a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(T(x), T(y)) \leq k\,(d(T(x), x) + d(T(y), y)), \ \ for\ all\ x, y \in X, \tag{2.5}$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. Replacing y by $T(x)$ in (2.5), we have

$$d(T(x), T^2(x)) \leq k\,(d(x, T(x)) + d(T(x), T^2(x)) \text{ for every } x \in X.$$

So, it must be the case that

$$d(T(x), T^2(x)) \leq r\, d(x, T(x)) \ \text{ for every } x \in X,$$

where $0 \leq r = \frac{k}{1-k} < 1$.

By an argument similar to that used above, we have if $y \neq T(y)$, then

$$inf\{d(x, y) + d(x, T(x)) : x \in X\} > \theta.$$

Applying Corollary 2.1 we have the desired conclusion.

Theorem 2.5. Let (X, d) be a complete cone metric space, P a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(T(x), T(y)) \leq k\,(d(T(x), y) + d(T(y), x)), \ \ for\ all\ x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. The proof obtained by the same techniques as used above.

Theorem 2.6. Let (X, d) be a complete cone metric space with a w-distance p and $\leq$ be a complete ordering on E with respect to P. Let T_1, T_2 be mappings from X onto itself. Suppose that there exists $r > 1$ such that

$$min\,\{\,p(T_2T_1(x), T_1(x)),\ p(T_1T_2(x), T_2(x))\} \geq r\, max\,\{\,p(T_1(x), x), p(T_2(x), x)\} \tag{2.6}$$

for every $x \in X$ and that

$$inf\,\{p(x, y) + min\,\{\,p(T_1(x), x), p(T_2(x), x)\} : x \in X\} > \theta \tag{2.7}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v, v) = \theta$.

Proof. Let u_0 be an arbitrary element of X. T_1 being onto, there exists an element u_1 satisfying $u_1 \in T_1^{-1}(u_0)$. Since T_2 is also onto, there is an element u_2 such that $u_2 \in T_2^{-1}(u_1)$. Proceeding in a similar way, we can find

$u_{2n+1} \in T_1^{-1}(u_{2n})$ and $u_{2n+2} \in T_2^{-1}(u_{2n+1})$ for $n = 1, 2, 3, \cdots$.

Therefore, $u_{2n} = T_1(u_{2n+1})$ and $u_{2n+1} = T_2(u_{2n+2})$ for $n = 0, 1, 2, \cdots$.

Using condition (2.6), we have for any positive integer n,

$$p(u_{n-1}, u_n) \geq r\, p(u_n, u_{n+1})$$

which implies that,

$$p(u_n, u_{n+1}) \leq \frac{1}{r} p(u_{n-1}, u_n) \leq \cdots \leq \left(\frac{1}{r}\right)^n p(u_0, u_1). \tag{2.8}$$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < 1$ since $r > 1$.

Now, (2.8) becomes

$$p(u_n, u_{n+1}) \leq \alpha^n p(u_0, u_1).$$

If $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq \left[\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}\right] p(u_0, u_1) \\ &\leq \frac{\alpha^n}{1-\alpha} p(u_0, u_1). \end{aligned}$$

But $\left(\frac{\alpha^n}{1-\alpha} p(u_0, u_1)\right)$ is a sequence in P converging to θ. So, by Lemma 2.1(iii), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in N$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \leq \lim_{m\to\infty} inf\, p(u_n, u_m) \leq \frac{\alpha^n}{1-\alpha} p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} \theta &< inf\, \{p(x, z) + min\, \{p(T_1(x), x), p(T_2(x), x)\} : x \in X\} \\ &\leq inf\, \{p(u_n, z) + min\, \{p(T_1(u_n), u_n), p(T_2(u_n), u_n)\} : n \in N\} \\ &\leq inf\, \left\{\frac{\alpha^n}{1-\alpha} p(u_0, u_1) + p(u_{n-1}, u_n) : n \in N\right\} \\ &\leq inf\, \left\{\frac{\alpha^n}{1-\alpha} p(u_0, u_1) + \alpha^{n-1} p(u_0, u_1) : n \in N\right\} \\ &= \theta \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

Suppose that $v = T_1(v) = T_2(v)$ for some $v \in X$. Then

$$\begin{aligned} p(v, v) &= min\, \{p(T_2T_1(v), T_1(v)), p(T_1T_2(v), T_2(v))\} \\ &\geq r\, max\, \{p(T_1(v), v), p(T_2(v), v)\} \\ &= r\, max\, \{p(v, v), p(v, v)\} \\ &= r\, p(v, v). \end{aligned}$$

By Proposition 1.1, we have $p(v, v) = \theta$.

Corollary 2.2. Let p be a w-distance on a complete cone metric space (X, d) and let $T : X \to X$ be an onto mapping. Suppose that there exists $r > 1$ such that

$$p(T^2(x), T(x)) \geq rp(T(x), x) \tag{2.9}$$

for every $x \in X$ and that

$$inf\{p(x, y) + p(T(x), x) : x \in X\} > \theta \tag{2.10}$$

for every $y \in X$ and that $y \neq T(y)$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v, v) = \theta$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.6, we have the desired result.

The following theorem is the generalization of the result [15; Theorem 3] to cone metric spaces.

Theorem 2.7. Let (X, d) be a complete cone metric space, P a normal cone and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying

$$d(T(x), T(y)) \geq r\, min\{d(T(x), x), d(T(y), y), d(x, y)\} \tag{2.11}$$

for any $x, y \in X$, and T is onto continuous, then T has a fixed point.

Proof. Since P is normal, d is a w-distance on X. Replacing y by $T(x)$ in (2.11), we obtain

$$d(T(x), T^2(x)) \geq r\, min\{d(T(x), x), d(T^2(x), T(x)), d(x, T(x))\} \tag{2.12}$$

for all $x \in X$.

We assume that $T(x) \neq T^2(x)$. Otherwise, T has a fixed point.

So, it follows from (2.12) that

$$d(T^2(x), T(x)) \geq rd(T(x), x)$$

for every $x \in X$.

Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\{d(x, y) + d(T(x), x) : x \in X\} = \theta.$$

Then, there exists a sequence (x_n) In X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(T(x_n), x_n)\} = \theta,$$

which gives that $d(x_n, y) \to \theta$ and $d(x_n, T(x_n)) \to \theta$. By Lemma 2.1(i), $T(x_n))$ converges to y. Using continuity of T, we have

$$T(y) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = y,$$

which is a contradiction.

Hence, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(T(x), x) : x \in X\} > \theta.$$

Thus, condition (2.10) is satisfied and Corollary 2.2 applies to obtain a fixed point of T.

Remark 2.1. For an expansive mapping $T : X \to X$, there exists $r > 1$ such that

$$d(T(x), T(y)) \geq r\, d(x, y) \geq r\ min\{d(T(x), x), d(T(y), y), d(x, y)\}$$

for all $x, y \in X$. However, the identity mapping satisfies condition (2.11) but it is not expansive. Thus, the class of mappings that considered in Theorem 2.7 is strictly larger than that of expansive mappings.

Theorem 2.8. Let (X, d) be a complete cone metric space, P a normal cone and the mapping $T : X \to X$ is continuous, onto and satisfies the condition

$$d(T(x), T(y)) \geq k\,[d(T(x), x) + d(T(y), y)] \qquad (2.13)$$

for all $x, y \in X$, where $\frac{1}{2} < k < 1$ is a constant. Then T has a fixed point in X.

Proof. Replacing x by $T(x)$ and y by x in (2.13), we have

$$d(T^2(x), T(x)) \geq k\,[d(T^2(x), T(x)) + d(T(x), x)]$$

which implies that

$$d(T^2(x), T(x)) \geq r\, d(T(x), x) \;\; for\ all\ x \in X,$$

where $r = \frac{k}{1-k} > 1$. By the same methods used above, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(T(x), x) : x \in X\} > \theta,$$

which is condition (2.10) of Corollary 2.2.
Applying Corollary 2.2 we obtain the desired conclusion.
The following is the generalization of Caristi's theorem[2] to cone metric spaces.

Theorem 2.9. Let p be a w-distance in a complete cone metric space (X, d), P a regular cone. Let T be a continuous mapping from X into itself. Suppose that there exists a mapping $Q : X \to P$ such that

$$p(x, T(x)) \leq Q(x) - Q(T(x))$$

for all $x \in X$. Then T has a fixed point in X. Moreover, if $v = T(v)$ then $p(v, v) = \theta$.

Proof. Let $u_0 \in X$ and let (u_n) be defined as follows:

$$u_n = T(u_{n-1}) = T^n(u_0) \;\; for\ n = 1, 2, 3, \cdots.$$

For any positive integer r, we have

$$\begin{aligned} p(u_r, u_{r+1}) &= p(u_r, T(u_r)) \\ &\leq Q(u_r) - Q(T(u_r)) \\ &= Q(u_r) - Q(u_{r+1}). \end{aligned}$$

Therefore,

$$\sum_{r=0}^{n-1} p(u_r, u_{r+1}) \leq \sum_{r=0}^{n-1} [Q(u_r) - Q(u_{r+1})] = Q(u_0) - Q(u_n) \leq Q(u_0).$$

Since P is regular, the series $\sum_{r=0}^{\infty} p(u_r, u_{r+1})$ is convergent.

If $m, n \in N$, $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &= \sum_{r=n}^{m-1} p(u_r, u_{r+1}). \end{aligned} \qquad (2.14)$$

Since the series $\sum_{r=0}^{\infty} p(u_r, u_{r+1})$ is convergent, by applying Lemma 2.1(iii), it follows from (2.14) that (u_n)

is Cauchy. By completeness of (X, d), there exists $v \in X$ such that $\lim_n u_n = v$.

Using continuity of T we have

$$T(v) = \lim_n T(u_n) = \lim_n u_{n+1} = v.$$

So, v is a fixed point of T.

Now,

$$p(v,v) = p(v, T(v)) \le Q(v) - Q(T(v)) = \theta$$

implies that $-p(v,v) \in P$.

Also, $p(v,v) \in P$. Since $P \cap (-P) = \theta$, we have $p(v,v) = \theta$.

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